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Theory of magnetic oscillations in Josephson-junction stacks

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Abstract

We consider magnetic oscillations of the critical current in stacks of intrinsic Josephson junctions. Depending on junction lateral size and magnetic field, oscillations may have either the period of half a flux quantum per junction (wide-stack regime) or one flux quantum per junction (narrow-stack regime). For junctions with lateral sizes of the order of several Josephson lengths, the stack crosses over from the wide-stack regime to the narrow-stack regime with increasing magnetic field. This crossover occurs via suppression of the critical-current peaks at the integer-flux-quanta points and enhancement of the critical-current peaks at the half-integer-flux-quanta points. In the narrow-stack regime the lattice structure periodically transforms between rectangular and triangular configurations. The latter configurations is realized only in narrow regions near magnetic-field values corresponding to an integer number of flux quanta per junction.
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1. Introduction

Intrinsic Josephson effect in layered high-temperature superconductors, such as $Bi_2Sr_2CaCu_2O_x$ (BSCCO), has been the subject of intense experimental and theoretical research in the past fifteen years, see reviews [1,2]. In a large-size superconductor the magnetic field applied along the layers generates a triangular lattice of Josephson vortices. When the magnetic field exceeds the crossover field $B_{cr} = \Phi_0/(2\pi\gamma s^2)$ ($\sim$0.5 T for BSCCO) the Josephson vortices homogeneously fill all layers [3]. Here γ is the anisotropy factor and s is the interlayer periodicity. Strong coupling between the vortex arrays in neighboring layers mediated by the in-plane supercurrents [4] determines the static and dynamic properties of the lattice. When an external transport current flowing across the layers exceeds the *critical current*, the Josephson vortex lattice starts to move. In homogeneous junctions the critical current is determined by interaction with the boundaries. For a single small Josephson junction without inhomogeneities the field dependence of the critical current is given by the celebrated Fraunhofer dependence, $J_c(\Phi) = J_J|\sin(\pi\Phi/\Phi_0)|/(\pi\Phi/\Phi_0)$, with Φ being the magnetic flux through the junction and J_J is the Josephson current. In other cases the critical current has more complicated field dependence.

The behavior of the critical current for the dense Josephson-vortex lattice in a homogeneous wide stack has been considered in Ref. [5]. The boundary induces an alternating deformation of the lattice decaying inside the superconductor at the typical length $2\sqrt{2}L_B$, which is larger than the Josephson length, $\lambda_J = \gamma s$, and increases proportional to the magnetic field, $L_B = \lambda_J B/B_{cr}$. The stack is in the large-size regime if its lateral width L is larger than this L_B. In this situation the surface deformation and the total current flowing along the surface is uniquely determined by the lattice displacement. The surface current has oscillating dependence on the lattice displacement and, due to the triangular-lattice ground state, the period of this

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dependence is half the lattice spacing. The total current flowing through the stack is given by the sum of two independent surface currents flowing at the sample edges. The magnetic field determines the magnitude of the maximum surface current and sets the phase shift between the oscillating dependences of the two surface currents on the lattice position. Due to the half-lattice-spacing periodicity of the surface current, a full-period change of this phase shift corresponds to the change of the magnetic flux through one junction, Φ, equal to the half flux quantum, $\Phi_0/2$. As a consequence, the maximum current through the stack has field dependence similar to the Fraunhofer dependence: it has strong oscillations and overall $1/B$ dependence. However, the period of these oscillations is two times smaller: it corresponds to adding *one flux quantum per two junctions* and the critical current has local maxima at $\Phi = j\Phi_0/2$. Oscillations of the flux-flow voltage caused by this effect have been observed by Ooi et al. [6], see also numerical simulations [7].

In this proceeding we extend our consideration to the regime when the junction size L becomes comparable with the size L_B and the system crosses over from the wide-stack to narrow-stack regime. As the length scale L_B increases with the magnetic field, it also sets the field scale $B_L = B_{cr}L/\lambda_J$, at which L_B becomes of the order of L. $[\Phi_0/2]$-periodicity of the critical-current oscillations holds until the surface deformations at the opposite edges are independent. Interactions between the surface deformations break the $\Phi_0/2$-periodicity: they lead to enhancement of the critical current at points $\Phi = (k + 1/2)\Phi_0$ and to suppression of the critical current at points $\Phi = k\Phi_0$. In the region $L \sim L_B$ the lattice structure is determined by a competition between the two energies: the interaction with boundaries and the bulk shearing interaction between the Josephson-vortex planar arrays in neighboring layers. The boundary interactions favor the aligned rectangular arrangement of the Josephson vortices while the local shearing interaction favors the triangular lattice. The boundary interactions decay slower with increasing field than the shearing interactions and become dominating at large fields. On the other hand, the boundary interaction energy has oscillating field dependence and vanishes at the points $\Phi = k\Phi_0$. At these points the shearing interaction is relevant even at $B > B_L$.

In the region $L \ll L_B$ the rectangular arrangement of vortices is realized in most part of the phase space and the field dependence of the critical current approaches the classical Fraunhofer dependence. Two important deviations persist at all fields and sizes: (i) Near the points $\Phi = k\Phi_0$ the phase transition to the triangular lattice always takes place. The critical current at these points never drops to zero and actually always has a small local maximum. (ii) Away from the points $\Phi = k\Phi_0$ the critical current is reached at the instability point of the rectangular vortex lattice and it is always somewhat smaller than the "Fraunhofer" value $I_{c0}|\sin(\pi\Phi/\Phi_0)|/|\pi\Phi/\Phi_0|$.

2. Smooth alternating lattice deformation for Josephson-junction stack at high magnetic field

We consider a Josephson-junction stack consisting of N layers with lateral size L in a magnetic field B applied along the layers. Below we briefly outline derivations and summarize the main results. Details will be published elsewhere. At high magnetic fields the junction properties are described by the reduced energy per one junction and per unit length in the field direction which we represent in the following form:

$$\mathscr{E} = \frac{1}{N}\sum_n \int_0^L \mathrm{d}u \left[\frac{1}{2}\left(\frac{\mathrm{d}\phi_n}{\mathrm{d}u}\right)^2 - \cos(\phi_{n+1} - \phi_n - hu + \alpha + \pi n)\right], \quad (1)$$

where ϕ_n are the in-plane phases and $h \equiv B/B_{cr}$ is the reduced magnetic field. We use the Josephson length λ_J as the unit of length and $E_J\lambda_J$ as the unit of energy per unit length, where E_J is the Josephson coupling energy per unit area. The parameter α describes lattice displacement. The oscillating behavior is determined by the reduced parameter hL which is directly related to the total magnetic flux through one junction $\Phi = BLs$, as $hL = 2\pi\Phi/\Phi_0$. The ratio L/h determines crossover between the wide-stack and narrow-stack regimes [5]. In the wide-stack regime $L \gg h$ away from the boundaries the phases ϕ_n are small and rapidly oscillate in space.

We will assume the alternating phase distribution in the form $\phi_n = (-1)^n\phi$ and split the alternating phase into the smooth and rapidly-oscillating components [5], $\phi(u) = v(u) + \tilde{\phi}(u)$, where we assume $|\tilde{\phi}| \gtrsim |\mathrm{d}v/\mathrm{d}u| \ll 1$. Averaging with respect to rapid oscillations, $\tilde{\phi}$, leads to the sine-Gordon equation for the smooth phase [5]

$$\frac{\mathrm{d}^2 v}{\mathrm{d}u^2} - \frac{2}{h^2}\sin(4v) = 0. \quad (2)$$

Using the boundary conditions $\mathrm{d}\phi/\mathrm{d}u = \mathrm{d}v/\mathrm{d}u + \mathrm{d}\tilde{\phi}/\mathrm{d}u = 0$ for $u = 0$, L and computing the derivatives of the oscillating phase at the edges, we also derive the boundary conditions for the smooth phase.

$$\frac{\mathrm{d}v}{\mathrm{d}u}(0) = \frac{2}{h}\cos(2v_0)\cos(\alpha), \quad (3)$$

$$\frac{\mathrm{d}v}{\mathrm{d}u}(L) = \frac{2}{h}\cos(2v_L)\cos(-hL + \alpha), \quad (4)$$

with $v_0 \equiv v(0)$ and $v_L \equiv v(L)$. The deformation field $v(u)$ determines static properties of the lattice. In particular, the energy in terms of the smooth phase $v(u)$ is given by

$$\mathscr{E}(h, L, \alpha) \approx \frac{1}{h}[\sin(2v_0)\cos(\alpha) - \sin(2v_L)\cos(-hL + \alpha)] + \int_0^L \mathrm{d}u \left[\frac{1}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2 - \frac{1 + \cos 4v}{2h^2}\right]. \quad (5)$$

The total Josephson current flowing through the stack is proportional to $d\mathscr{E}_J/d\alpha$ and is given by

$$J(h,L,\alpha) = \frac{1}{h}[-\sin(2v_0)\sin\alpha + \sin(2v_L)\sin(-hL+\alpha)]. \tag{6}$$

The unit of current here is $j_J\lambda_J w$ where j_J is the maximum Josephson-current density and w is the junction size in the field direction.

A general solution of the sine-Gordon equation (2) can be found in terms of elliptic integrals,

$$\int_{v_0}^{v} \frac{dv}{\sqrt{1/m - \cos^2(2v)}} = \pm\sqrt{2}u/h. \tag{7}$$

The shape of the solution is determined by the elliptic-integral parameter m. Using the equation for $v(u)$ the boundary conditions can be transformed to the following form:

$$\cos(2v_0) = \sqrt{\frac{1/m}{1+2\cos^2(\alpha)}}, \tag{8}$$

$$\cos(2v_L) = \sqrt{\frac{1/m}{1+2\cos^2(hL-\alpha)}}. \tag{9}$$

Depending on parameters α and hL, solutions may be either monotonic [$\cos\alpha\cos(hL-\alpha) > 0$] or nonmonotonic [$\cos\alpha\cos(hL-\alpha) < 0$]. For these two cases one can derive from Eq. (7) the following equation connecting m with v_0 and v_L:

$$\sqrt{m}\left[F\left(\frac{\pi}{2}+2v_L,m\right) - F\left(\frac{\pi}{2}+2v_0,m\right)\right] = \mathrm{sign}[\cos\alpha]\sqrt{8}L/h \tag{10}$$

for the monotonic solution and

$$\sqrt{m}\left[2F\left(\arcsin\left(\frac{1}{\sqrt{m}}\right),m\right) - F\left(\frac{\pi}{2}+2v_0,m\right) - F\left(\frac{\pi}{2}+2v_L,m\right)\right] = \pm\frac{\sqrt{8}L}{h} \tag{11}$$

for the nonmonotonic solution, where $F(\varphi,m) = \int_0^{\varphi} dx(1-m\sin^2 x)^{-1/2}$ is the incomplete elliptic integral of the first kind [9]. Eqs. (8)–(10) [or (11)] have to be solved for given L, h, and α to find the three unknown constants v_0, v_L, and m, which completely determine the solution and lattice properties. Ground states are realized at symmetric lattice locations, corresponding to either $\alpha = hL/2 + \pi k$ or $\alpha = hL/2 + \pi/2 + \pi k$. The critical current is given by the maximum of $J(h,L,\alpha)$ given by Eq. (6) with respect to α.

3. Field evolution of lattice structure and critical current

At large L, $L \gg h$, the parameter m is close to unity and the smooth alternating deformation has solutions in the form of two isolated surface solitons [5]. The monotonic solution corresponds to solitons of the same sign and the nonmonotonic solution corresponds to solitons with the opposite signs. If one neglects interaction between the solitons then the relative sign of the surface solitons has no importance and the total Josephson current is given by the sum of two independent surface currents, which do not depend on the soliton signs [5]. As a consequence, the product hJ_c has periodicity of $\Phi_0/2$ per junction. At finite L the interaction between the surface solitons disturbs such periodicity. At the integer-flux-quanta points $\Phi = k\Phi_0$ the surface solitons have the same sign and repel each other. As a consequence, the amplitude of surface deformations drops and the critical current decreases. At the half-integer-flux-quanta points $\Phi = (k+1/2)\Phi_0$ the situation is opposite: the surface solitons have opposite signs and attract each other leading to enhancement of the surface deformations and increase of the critical current.

In the region $L/h < 1$, a transition to the rectangular lattice takes place, corresponding to the limits $v_{0,L} \to \pm\pi/4$ and $m \to \infty$. One can derive a general conditions for this state at arbitrary h and α. In particular, the rectangular lattice gives a local energy minimum at $\alpha = hL/2 + \pi/2$ in the regions $|\Phi/\Phi_0 - (k+1/2)| < 1/4$ if the inequality

$$|\sin(hL/2)| < \tan(\sqrt{2}L/h)/\sqrt{2}$$

is satisfied. The rectangular lattice first appears in the ground state at points $\Phi = (k+1/2)\Phi_0$ for $L/h \leqslant l_1 = \arctan(\sqrt{2})/\sqrt{2} \approx 0.675$. If L/h is only slightly smaller than l_1 then the rectangular lattice becomes unstable with increasing current. When L/h drops below $l_2 \approx 0.484$, the rectangular lattice remains stable at $\Phi = (k+1/2)\Phi_0$ up to the critical current. The rectangular lattice is never realized at the points $\Phi = k\Phi_0$. Regions of ground-state rectangular lattice in the plane size-magnetic field are shown in Fig. 1. We also show three representative lattice configurations in the crossover region for $L = 4$.

We systematically investigated the evolution of the field dependence of the critical current with changing stack size L. Fig. 2 shows the field dependences of the critical current for two junction sizes, 2.5 and 4. For the [$L = 4$]-plot one can observe that with increasing field [$\Phi_0/2$]-periodic oscillations smoothly transform into Φ_0-periodic oscillations. This occurs via suppression of the peaks at $\Phi = k\Phi_0$ and enhancement of the peaks at $\Phi = (k+1/2)\Phi_0$. At larger L the crossover takes place at larger field and larger Φ/Φ_0. We also show regions of stable rectangular lattice in the field-current plane. One can see that in both cases the rectangular lattice first appears when h/L exceeds $1/l_1$ and its stability range extends up to the critical current when h/L exceeds $1/l_2$. At slow lattice motion in the overdamped regime the flux-flow voltage has similar field oscillations with amplitude that roughly scales as $U_{ff}J_c^2/J_{ext}^2$, where U_{ff} is the flux-flow voltage and J_{ext} is the applied transport current [5]. The crossover between the wide-stack and narrow-stack regimes in the flux-flow voltage oscillations has been recently experimentally studied by Kakeya et al. [8].

In the region $L/h \ll 1$ the rectangular lattice occupies most part of the phase space and the field dependence of

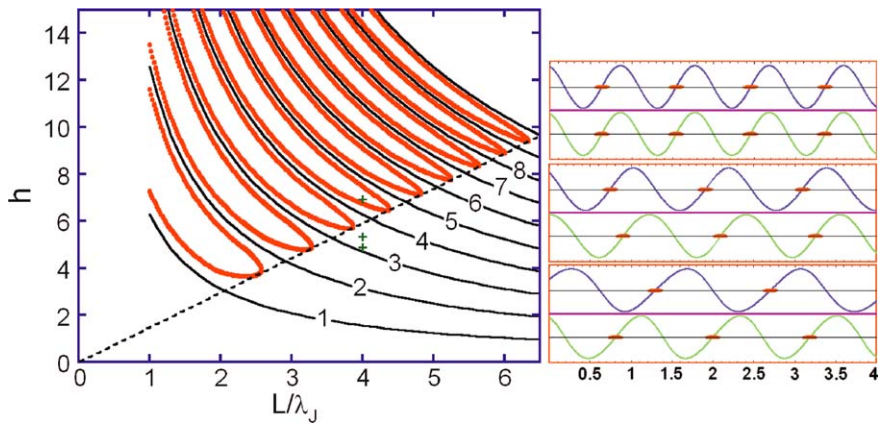

Fig. 1. *Left plot* shows the phase diagram of the Josephson-junction stack in the coordinates [junction size L/λ_J]–[magnetic field $h = B/B_{cr}$]. The solid lines mark fields corresponding to the integer values of the magnetic flux per junctions, $\Phi = k\Phi_0$. The thick dotted lines show boundaries of the the rectangular-lattice regions in the ground state. The dashed line $h = L/l_1$ marks the crossover between the narrow-stack and wide-stack regimes. *Right plots* show representative ground state lattice structures at three marked points corresponding to values $\Phi/\Phi_0 = 4.4$, 3.4, and 3.1 (from top to bottom). Lines show oscillating z-axis currents in neighboring layers and ellipses mark centers of the Josephson vortices.

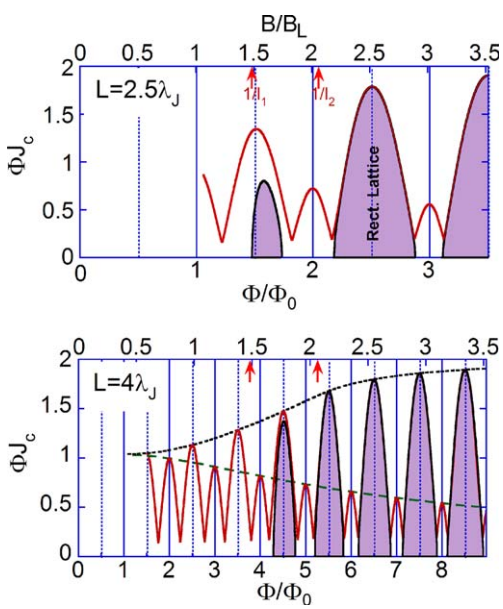

Fig. 2. The field dependences of critical current (solid lines) for stack sizes $L = 2.5$ and 4 for the same range of the ratio $B/B_L \equiv h/L$ shown on the top axes. To emphasize the periodic nature of these dependences, we plot the product ΦJ_c (in units of $J_J\Phi_0/2\pi$ where J_J is the total maximum Josephson current) vs Φ/Φ_0. Shaded areas show the regions of stable rectangular lattice. The rectangular-lattice regions appear in the vicinity of points $\Phi = (k + 1/2)\Phi_0$ when h/L exceeds $1/l_1 \approx 1.48$. When h/L exceeds $1/l_2 = 2.07$ the rectangular lattice remains stable up to the critical current.

the critical current approaches the Fraunhofer dependence. The triangular lattices exist only in the narrow vicinity of the points $\Phi = k\Phi_0$. In the ground state the transformation between the triangular and rectangular states occurs via a first-order transition at the flux values, Φ_t given by the following equation:

$$\frac{2\pi\Phi_t}{\Phi_0}\left|\sin\left(\frac{\pi\Phi_t}{\Phi_0}\right)\right| = \frac{3}{2}\left(\frac{L}{\lambda_J}\right)^2. \tag{12}$$

We also found that, in contrast to the Fraunhofer dependence, the critical current at $\Phi = k\Phi_0$ does not drop to zero and even has a small peak. The value of J_c at the peak maximum decays with Φ as

$$J_c(\Phi = k\Phi_0) = \frac{J_J}{(2\pi^2)}\left(\frac{\Phi_0}{\Phi}\right)^2\left(\frac{L}{\lambda_J}\right)^2.$$

On the other hand, the critical current at $\Phi = (k + 1/2)\Phi_0$ is achieved at the instability point of the rectangular lattice and it is close to the "Fraunhofer" value $J_c = J_J\Phi_0/(\pi\Phi)$.

In summary, we studied the magnetic field evolution of the ground-state lattice configurations and critical current in the intrinsic Josephson-junction stacks with lateral size larger or comparable with the Josephson length. With increasing magnetic field the period of the critical-current oscillations evolves from $\Phi_0/2$ per junction to Φ_0 per junction. This crossover takes place when the magnetic field reaches the typical value B_L proportional to the junction lateral size. At $B > B_L$ the lattice has rectangular configuration in most part of the phase space and the field dependence of the critical current is close to the Fraunhofer dependence. The only exception are narrow regions in the vicinity of the points $\Phi = k\Phi_0$ where the lattice transforms to the triangular state via a first-order phase transition. Due to this transition, the critical current does not vanish at these points and even has small peaks.

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