Microscopic Theory of Thermal Phase Slips in Clean Narrow Superconducting Wires

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We consider the structure of a thermal phase-slip center for a simple microscopic model of a clean one-dimensional superconductor in which superconductivity occurs only within one conducting channel or several identical channels. Surprisingly, the Eilenberger equations describing the saddle-point configuration allow for an exact analytical solution in the whole temperature and current range. This solution allows us to derive a closed expression for the free-energy barrier, which we use to compute its temperature and current dependences.

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Recent advance in fabrication techniques have brought ultrathin superconductor wires into active experimental investigation (for example, [1–4]), as they may play an important role in future electronics. These studies brought about the fundamental question of the limits at which thin wires retain a key property of superconductors: the ability to carry currents without dissipation. It was well established that in extremely narrow superconductors thermal fluctuations can occasionally drive a small part of the wire into a normal state. When there is only one channel for a supercurrent to pass, these fluctuations interrupt its flow and cause a finite effective resistance. This phenomenon has been understood by Langer and Ambegaokar (LA) as thermally activated *phase slips* [5]: local abrupt switches of the system between its metastable states in which the event phase jumps discontinuously by $\pm 2\pi$; see also Ref. [6] for a review. The corresponding resistance exhibits Arrhenius law temperature behavior, $\rho \propto \exp(-\delta F/T)$, where δF is the energy barrier separating these states. In a phase-slip event the order parameter goes through the saddle-point configuration in which it changes sign as a function of the coordinate and vanishes at one point. The barrier for the phase slip, δF, is given by the energy of this saddle-point configuration. Later, a preexponential factor in resistivity has been calculated by McCumber and Halperin [7] and Langer-Ambegoakar-McCumber-Halperin (LAMH) theory has been confirmed experimentally on tin whiskers [8]. The dynamical theory of phase slips at large currents has been further extended in Refs. [9,10].

The mechanism of appearance of a finite resistivity in a thin wire can be described as follows [6]. A finite voltage drop across the wire means that the order parameter phase difference ϕ_{12} between the ends of the wire has to increase linearly with time. At first glance, this should inevitably lead to a continuous increase of the supercurrent flowing through the wire, which is obviously inconsistent with the steady state. The resolution of this apparent inconsistency is that the current relaxes via the phase-slip events, in which the order parameter vanishes at some point in the superconductor, allowing a phase difference between opposite sides of this point to change by 2π. As a result of this event, a phase jump equal to 2π appears at one point of the superconductor reducing the phase gradient in the rest of the wire. To maintain a steady state, such events must occur with the average frequency $2e\langle V\rangle/\hbar$.

Phase-slip events can be caused also by quantum tunneling, which at sufficiently low temperatures can prevail over the thermal activation, and this possibility is being widely searched and debated [1–4,11]. Yet, the role and the very fact of observation of quantum phase slips remain a controversy. It seems now plausible that thermally activated phase slips may dominate down to very low temperatures and that quantum fluctuations do not add any significant corrections to the resistance in almost the whole experimentally accessible temperature range. As original LAMH theory [5,7] was developed within the framework of the Ginzburg-Landau (GL) approach, it holds, strictly speaking, only in the temperature interval near the critical temperature, T_c, of the superconductor transition. On the other hand, in sufficiently narrow wires a finite thermally activated resistivity can be observed down to temperatures which are significantly lower than T_c, There is thus an experimental quest for extension of the LAMH theory to low temperatures, where the phase slips have to be analyzed within the microscopic theory.

In this Letter we investigate a simple model of a clean one-dimensional superconductor containing either only one conducting channel or a few degenerate channels. It turns out that this model allows for full analytical treatment of a free-energy saddle-point configuration. An exact analytical solution of the Eilenberger equation is found, both at zero and finite supercurrent. This solution allows us to derive a closed expression for the temperature and current dependent free-energy barrier.

We consider the simplest case of a narrow clean superconducting wire containing either only one superconducting channel or several degenerate channels, i.e., with the same Fermi velocity, v_F. The wire diameter is assumed to be smaller than the London penetration depth; one can thus neglect the magnetic field generated by the currents. We

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will find an order parameter configuration which corresponds to the free-energy saddle point in such a wire. As the saddle-point configuration is a stationary point in the parameter space, it is described by the same equations as the equilibrium state. In the quasiclassical approximation, such a superconducting wire is described by the gap parameter $\Delta(x)$ and the Green's functions, $g, f, f^\dagger$ depending on the coordinate along the wire x, the Matsubara frequencies $\omega_n = 2\pi T(n + 1/2)$, and the location on the Fermi surface, which is reduced to two points corresponding to forward and back motion. The Green's functions are connected by the normalization condition $g^2 + ff^\dagger = 1$ and obey the one-dimensional Eilenberger equations [12]:

$$v_F \frac{\partial g}{\partial x} + \Delta f^\dagger - \Delta^* f = 0, \tag{1a}$$

$$-v_F \frac{\partial f}{\partial x} - 2\omega_n f + 2\Delta g = 0, \tag{1b}$$

$$v_F \frac{\partial f^\dagger}{\partial x} - 2\omega_n f^\dagger + 2\Delta^* g = 0, \tag{1c}$$

with $v_F = \pm|v_F|$. It is convenient to introduce new dimensionless variables

$$x \to x/\xi, \qquad \Delta \to \Delta/T_c, \qquad T \to T/T_c,$$

where $\xi = v_F/T_c$ is the coherence length and T_c is the critical temperature. In new variables the Eilenberger equations for the forward direction, $v_F > 0$, become

$$\frac{\partial g}{\partial x} + \Delta f^\dagger - \Delta^* f = 0, \tag{2a}$$

$$-\frac{\partial f}{\partial x} - 2\omega_n f + 2\Delta g = 0, \tag{2b}$$

$$\frac{\partial f^\dagger}{\partial x} - 2\omega_n f^\dagger + 2\Delta^* g = 0. \tag{2c}$$

These equations must be completed with the self-consistency equation,

$$\frac{\Delta(x)}{\lambda} = \pi T \sum_{\omega_n} \langle f(\omega_n, x, \hat{k}_F)\rangle_F, \tag{3}$$

and the formula for the supercurrent,

$$j = -2\pi i e N(0) T \sum_{\omega_n} \langle v_F g(\omega_n, x, \hat{k}_F)\rangle_F, \tag{4}$$

where $N(0) = 1/(\pi v_F)$ is the density of states. The summation is going over all Matsubara frequencies $n = -\infty \ldots \infty$. For the one-dimensional problem averaging over the Fermi surface, $\langle \ldots \rangle_F$, reduces to summation over two points

$$\langle f(\omega_n, x, \hat{k}_F)\rangle_F = \frac{1}{2}[f(\omega_n, x, k_F) + f(\omega_n, x, -k_F)],$$

$$\langle v_F g(\omega_n, x, \hat{k}_F)\rangle_F = \frac{v_F}{2}[g(\omega_n, x, k_F) - g(\omega_n, x, -k_F)].$$

The Green's functions in the opposite directions are connected by relations $f(-k_F) = [f^\dagger(k_F)]^*$, $f^\dagger(-k_F) = [f(k_F)]^*$, and $g(-k_F) = [g(k_F)]^*$.

Let a finite current flow through the wire. Then the order parameter, Δ, and the anomalous Green's functions f and $f^\dagger$ can be represented in the form of plane waves with slowly varying in space amplitudes, $\Delta, f \propto e^{ikx}$; $\Delta^*, f^\dagger \propto e^{-ikx}$, where the wave vector k is determined by the supercurrent. Defining new variables $f_\pm = (f \pm f^\dagger)/2$, one rewrites Eqs. (2a)–(2c) as

$$\frac{\partial g}{\partial x} + 2i\Delta_I f_+ - 2\Delta_R f_- = 0, \tag{5a}$$

$$\frac{\partial f_-}{\partial x} - 2\Delta_R g + 2\omega_n' f_+ = 0, \tag{5b}$$

$$\frac{\partial f_+}{\partial x} - 2i\Delta_I g + 2\omega_n' f_- = 0, \tag{5c}$$

with $\Delta_R = \mathrm{Re}\Delta$, $\Delta_I = \mathrm{Im}\Delta$, and $\omega_n' = \omega_n + ik/2$. Using relations between the Green's functions for the opposite directions, we can rewrite the self-consistency condition (3) separately for the real and imaginary parts of Δ as

$$\frac{\Delta_R(x)}{\lambda} = \pi T \sum_{\omega_n} \mathrm{Re} f_+(\omega_n, x), \tag{6a}$$

$$\frac{\Delta_I(x)}{\lambda} = \pi T \sum_{\omega_n} \mathrm{Im} f_-(\omega_n, x). \tag{6b}$$

Homogeneous solution for superconductor in equilibrium is well known,

$$f_+^{(0)} = S_n \Delta_{R0}, \qquad f_-^{(0)} = iS_n \Delta_I, \qquad g^{(0)} = S_n \omega_n' \tag{7}$$

with

$$S_n = 1/\sqrt{\omega_n'^2 + |\Delta_0|^2}. \tag{8}$$

The absolute value of the equilibrium order parameter and supercurrent can be found from Eqs. (3) and (4) as

$$\pi T \sum_{\omega_n} \left(\mathrm{Re} \frac{1}{\sqrt{\omega_n'^2 + |\Delta_0|^2}} - \frac{1}{|\omega_n|} \right) = \ln T, \tag{9}$$

$$j = -2eT \sum_{\omega_n} \mathrm{Im} \frac{\omega_n'}{\sqrt{\omega_n'^2 + |\Delta_0|^2}}. \tag{10}$$

Note that the depairing action of superfluid velocity in our model is equivalent to the depairing due to the Zeeman splitting; see, e.g., Ref. [13]. As a consequence, in our system the temperature dependence of the critical current is nonmonotonic, similar to the Zeeman-effect upper critical field. Such temperature dependence is a peculiar property of clean one-dimensional superconductors and it is very different from higher dimensional systems.

We will demonstrate now that Eqs. (5) and (6) allow for an exact analytical solution. First, from Eqs. (5) we obtain a useful relation between the Green's functions

$$i\Delta_I \frac{\partial f_-}{\partial x} - \Delta_R \frac{\partial f_+}{\partial x} - \omega_n' \frac{\partial g}{\partial x} = 0. \tag{11}$$

In general, we cannot reduce this relation to the first integral of Eqs. (5a)–(5c). To proceed, we make some strong assumptions about the form of the order parameter and Green's functions. We will check consistency of these assumptions from the obtained solution. Namely, we assume that (i) the imaginary part of the order parameter Δ_I is coordinate independent and (ii) f_+ can be found in the variable-separated form

$$f_+(\omega_n, x) = S_n \Delta_R(x), \tag{12}$$

where the frequency term S_n is assumed to be coordinate independent and therefore it has to be given by Eq. (8). In this case we can rewrite Eq. (11) as a first integral,

$$\frac{\partial}{\partial x}\left[i\Delta_I f_- - S_n \frac{\Delta_R^2}{2} - \omega_n' g \right] = 0,$$

which gives $i\Delta_I f_- - S_n \frac{\Delta_R^2}{2} - \omega_n' g = \mathrm{C}$. The constant C can be found using asymptotics far away from the phase-slip center. Substituting for Green's functions their equilibrium values from Eq. (7), we obtain $\mathrm{C} = -S_n(\omega_n'^2 + \Delta_I^2 + \Delta_{R0}^2/2)$, which gives

$$i\Delta_I f_- - \omega_n' g = -S_n\left(\omega_n'^2 + \Delta_I^2 + \frac{\Delta_{R0}^2 - \Delta_R^2}{2}\right). \tag{13}$$

Taking derivative of Eq. (5c) and excluding $\partial g/\partial x$ and $\partial f_-/\partial x$ from Eqs. (5a) and (5b), we obtain

$$\frac{\partial^2 f_+}{\partial x^2} - 4(\Delta_I^2 + \omega_n'^2) f_+ - 4\Delta_R(i\Delta_I f_- - \omega_n' g) = 0. \tag{14}$$

Substituting for f_+ the presentation (12) and using the relation (13), we see that the frequency part indeed drops out and this equation reduces to the differential equation for Δ_R,

$$\frac{\partial^2 \Delta_R}{\partial x^2} + 2\Delta_{R0}^2 \Delta_R - 2\Delta_R^3 = 0, \tag{15}$$

which essentially coincides with the Ginzburg-Landau equation. However, in our case it describes the phase-slip center in the whole temperature range. This equation has a kink solitonlike solution

$$\Delta_R(x) = \Delta_{R0} \tanh(\Delta_{R0} x). \tag{16}$$

Corresponding Green's functions are

$$f_+ = S_n \Delta_{R0} \tanh(\Delta_{R0} x), \tag{17a}$$

$$f_- = S_n\left(i\Delta_I - \frac{\Delta_{R0}^2}{2(\omega_n' + i\Delta_I)} \frac{1}{\cosh^2(\Delta_{R0} x)}\right), \tag{17b}$$

$$g = S_n\left(\omega_n' + \frac{\Delta_{R0}^2}{2(\omega_n' + i\Delta_I)} \frac{1}{\cosh^2(\Delta_{R0} x)}\right). \tag{17c}$$

One can verify that the normalization constraint $g^2 + f_+^2 - f_-^2 = 1$ is satisfied. The assumed coordinate independence of the imaginary part of Δ requires that the coordinate-dependent part of Eq. (6b) vanishes. This gives the following equation

$$\sum_{\omega_n} \mathrm{Im}\left[\frac{1}{\omega_n' + i\Delta_I} \frac{1}{\sqrt{\omega_n'^2 + |\Delta_0|^2}}\right] = 0, \tag{18}$$

which can be used to find Δ_I, while $|\Delta_0|$ is determined by Eq. (9) and $\Delta_{R0} = \sqrt{|\Delta_0|^2 - \Delta_I^2}$. Therefore, the structure of the phase-slip center is fully defined.

The probability of a phase-slip event driven by thermal fluctuation is proportional to the exponent $\exp(-\delta F/T)$, where δF is the free-energy barrier between homogeneous and saddle-point state and T is the temperature. Therefore, the next point is to find the free-energy barrier.

The formula expressing the free-energy in terms of the quasiclassical Green's functions has been derived by Eilenberger in his original paper [12]. This formula can be straightforwardly generalized to the one-dimensional case and in real units it can be written as

$$F = N(0)\int dx\left(\frac{|\Delta|^2}{\lambda} - \pi T \sum_{\omega_n}\left\langle \frac{v_F g}{2}\left(\frac{\partial_x f}{f} - \frac{\partial_x f^\dagger}{f^\dagger}\right) + \Delta f^\dagger + \Delta^* f + 2\omega_n(g-1)\right\rangle_F\right). \tag{19}$$

The phase-slip barrier is given by the energy difference between homogeneous and inhomogeneous states, $\delta F = F - F_{\mathrm{hom}}$, and the free-energy density of a homogeneous state is known to be

$$\mathcal{F}_{\mathrm{hom}} = |\Delta|^2/\lambda - \pi T \sum_{\omega_n}\left(\sqrt{\omega_n^2 + |\Delta|^2} - |\omega_n|\right). \tag{20}$$

Using the Eilenberger equations and constraint $g^2 + ff^\dagger = 1$, we can simplify the expression for free-energy (19) and rewrite it without space derivatives. With the self-consistency equation we can get rid of the coupling constant λ. Subtracting the homogeneous-state free-energy from the full expression and using dimensionless units, we obtain the free-energy in units of T_c,

$$\delta F = \int \delta\mathcal{F} dx \tag{21}$$

with

$$\delta\mathcal{F} = \pi^{-1}(|\Delta|^2 - |\Delta_0|^2)\ln T + T\mathrm{Re}\sum_{\omega_n}\left[\frac{|\Delta|^2 - |\Delta_0|^2}{|\omega_n|} - \left(\frac{\Delta}{f} + \frac{\Delta^*}{f^\dagger} - \frac{2}{S_n}\right)\right]. \tag{22}$$

Using our saddle-point solution (16) and (17) and performing integration over x, we obtain for δF,

$$\delta F = T\mathrm{Re}\sum_{\omega_n}\left(\ln\frac{|\Delta_0|^2 - i\Delta_I\omega_n' + \Delta_{R0}\sqrt{\omega_n'^2 + |\Delta_0|^2}}{|\Delta_0|^2 - i\Delta_I\omega_n' - \Delta_{R0}\sqrt{\omega_n'^2 + |\Delta_0|^2}} - \frac{2\Delta_{R0}}{\sqrt{\omega_n'^2 + |\Delta_0|^2}}\right). \tag{23}$$

This formula gives the barrier in terms of the sum over

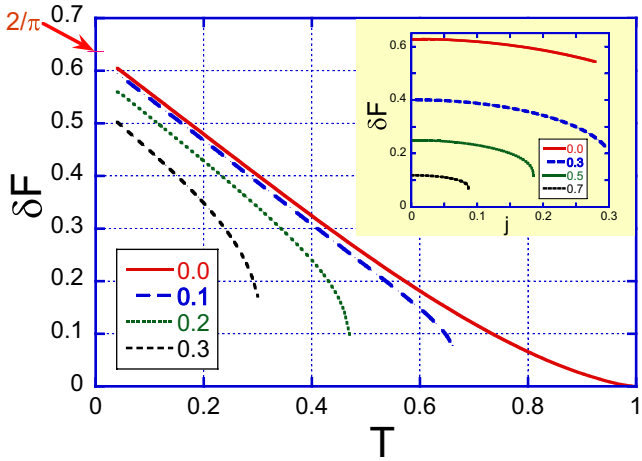

FIG. 1 (color online). The temperature dependences of the phase-slip barrier in units of $\Delta(0)$ for different currents. The legend indicates values of supercurrent measured in units of $j_0 = 2eT_c$. Note that at finite current the barrier always jumps to zero from a finite value. Inset shows current dependences of the barrier at different temperatures.

Matsubara frequencies, which allows for analytical calculation in the limits $T \to 0$, 1 and can be easily computed numerically in general case.

At zero-temperature the sum over Matsubara frequencies in (23) becomes an integral $T\sum_{\omega_n} \to \int d\omega/2\pi$. At $T \to 0$ the gap parameter $|\Delta_0|$ is given by the BCS value, $\Delta(0) = (\pi/\gamma)T_c$ with $\gamma = 1.7811$ and in our clean case it remains current-independent up to the critical current corresponding to $k \approx 0.83$. Equation (9) gives $\Delta_I \to -k/2$ for $T \to 0$. Integration over ω in Eq. (23) can be evaluated and leads to a very simple exact result for the zero-temperature barrier

$$\delta F(0, k) = \frac{2}{\pi}\Delta_{R0} = \frac{2}{\pi}\Delta(0)\sqrt{1 - \left(\frac{k}{2}\right)^2} \qquad (24)$$

for $k < 0.83$.

We also can find the asymptotics of the barrier in the Ginzburg-Landau region, $|T - T_c| \ll 1$, where our results have to be similar to the LAMH solution. In this temperature range $\Delta \ll 1$ and we can take into account only the first nonvanishing term of the Taylor expansion of Eq. (23). In this limit for a wire without current we have

$$\delta F = C_0\Delta(0)(1 - T)^{3/2}, \qquad (25)$$

where $C_0 = \frac{8\sqrt{2}}{3\sqrt{7\zeta(3)}}\frac{\gamma}{\pi} \approx 0.737$ with $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ as the Riemann ζ-function, $\zeta(3) \approx 1.202$.

For arbitrary temperatures and currents the free-energy barrier cannot be represented in a simple form and has be computed numerically. Figure 1 shows the temperature dependences of the barrier in units of $\Delta(0)$ for different dimensionless currents in units of $j_0 = 2eT_c$. We also show in the inset the current dependences of δF at several temperatures. One can see that the temperature dependences of the barrier have almost linear part at $T \to 0$. This is a consequence of the superconducting gap vanishing at one point. It is well known that in fully gapped superconductors the temperature dependences of all parameters are exponential. Accidently, the Ginzburg-Landau asymptotics (25) actually provides a reasonable approximation for the exact barrier in the whole temperature range. Even at $T \to 0$ the GL result exceeds the exact one only by 15%.

Our results can be straightforwardly generalized for a case of several identical conducting channels. If we have M channels with the same Fermi velocity, then the corresponding free-energy barrier is just proportional to the number of channels, $\delta F_M = M\delta F$. The current scale in Fig. 1 also scales with the number of channels, $j_0 \to j_M = 2MeT_c$. It is clear that the thicker the wire is and the more conducting channels it has, the higher the free-energy barrier is.

In conclusion, we considered the model of thermally activated phase slips in a clean one-dimensional superconductor with only one conducting channel and found the exact analytical solution of Eilenberger equations corresponding to the free-energy saddle point. This solution allowed us to calculate the temperature and current dependences of the free-energy barrier for thermal phase slips.

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