Correlation functions for an elastic string in a random potential: Instanton approach

Ya. M. Blanter[1,2] and V. M. Vinokur[2]
[1]*Department of Applied Physics and DIMES, Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands*
[2]*Materials Science Division, Argonne National Laboratory, Argonne, Illinois 60439*
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We develop an instanton technique for calculations of correlation functions characterizing statistical behavior of the elastic string in disordered media and apply the proposed approach to correlations of string free energies corresponding to different low-lying metastable positions. We find high-energy tails of correlation functions for the case of long-range disorder (the disorder correlation length well exceeds the characteristic distance between the sequential string positions) and short-range disorder, with the correlation length much smaller then the characteristic string displacements. The former case refers to energy distributions and correlations on the distances below the Larkin correlation length, while the latter describes correlations on the large spatial scales relevant for the creep dynamics.

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An elastic string in two-dimensional random potential is an archetypal problem of statistical physics, with applications to a widest variety of systems and phenomena. The examples include vortices in superconductors,[1] dislocations,[2] domain walls,[3] and non-Hermitian quantum mechanics.[4] The string that energetically disfavors overhangs (like vortices) maps onto *directed polymer*.[5] In the random potential the string bends to adjust itself to the pinning potential relief. Directed polymer approximation corresponds to *weak pinning*, where the string shape is smooth and coincides with the path of the classical particle traveling through the corresponding rugged energy landscape. The Hamiltonian of the string has the form[6]

$$H=\int dx\left[\frac{\kappa}{2}\left(\frac{d\zeta}{dx}\right)^2+V(x,\zeta)\right], \quad \left|\frac{\partial\zeta}{\partial x}\right|\ll 1. \tag{1}$$

Here x is the coordinate along the preferential direction (for small displacements it also measures the length of the segment), $\zeta(x)$ is transverse displacement, κ is the elastic constant, and the random potential $V(x,\zeta)$ is routinely assumed to be Gaussian, with zero average $\langle V\rangle=0$ and the correlation function

$$\langle V(x,\zeta)V(x',\zeta')\rangle=\beta\delta(x-x')K(\zeta-\zeta'), \quad \int K(y)dy=1.$$

The amplitude β depends on the type of disorder. For instance, for a vortex pinned in a plane by point defects, $\beta\approx\xi_0^2 n_i V^2$, with ξ_0, n_i, and V being the core radius, concentration of defects, and the energy of vortex-defect interaction, respectively.

Much of the early effort was concentrated on the fluctuations of the free energy of the string. Let the left end of the string be fixed at $\zeta=0$, $x_L<0$. The quantity of interest is then the free energy ε of the string as the function of the position of its right end (x,ζ). Knowledge of the mean free energy $\langle\varepsilon\rangle$ is important for calculation of dynamical quantities, such as pinning energy or drift velocity. Various tools, which, among others, include numerical analysis,[6] Bethe ansatz solution,[7] and power counting based on the effective action,[8] yield the result for the disorder-averaged free energy $\langle\varepsilon\rangle\propto(x-x_L)^{1/3}$ and the average displacement of the string $\langle\zeta\rangle\propto(x-x_L)^{2/3}$.

The dynamic response, however, requires more detailed statistical description, since in such a complex system, a non-self-averaging behavior can be expected. Many of the results have been already made available in the course of 15-year-long research, see Ref. 5 for an excellent review. For recent developments, we cite calculations of the free-energy cumulants[9] and tails of its distribution function.[10]

So far, attention has been mainly drawn to statistical properties of the free energy for a fixed position of the end, $\varepsilon(x,\zeta)$. The correlation functions describing different positions of the string, important for its slow dynamics (noise and velocity correlations), are also of interest. Conceptually, they also test the non-Gaussian shape of distributions. In this paper, we make the first step in this direction and evaluate the distribution function

$$P(u)=\langle\delta[u-\varepsilon(0,\zeta_1)+\varepsilon(0,\zeta_2)]\rangle, \tag{2}$$

which describes correlation between the sequential energies of the string, as it is displaced transversely to the preferential direction. In what follows, we assume $\zeta_1\geqslant\zeta_2$.

I. MODEL

The starting point of our theory is the equation for the free energy ε, which can be derived by the transfer-matrix method from Eq. (1), and reads

$$\frac{\partial\varepsilon}{\partial x}+\frac{1}{2\kappa}\left(\frac{\partial\varepsilon}{\partial\zeta}\right)^2-\frac{T}{2\kappa}\frac{\partial^2\varepsilon}{\partial\zeta^2}=V(x,\zeta), \tag{3}$$

where T is temperature. Equation (3) is easily recognized as Kardar-Parisi-Zhang equation in $1+1$ dimensions.[5] To this end, x and ζ can be identified as time and coordinate, respectively.

We focus on the behavior of the correlation function (2) at large u. In this case, $P(u)$ is the probability that the energy changes considerably upon small displacement of the string.

This probability is expected to be exponentially small, and an appropriate technique to calculate exponential tails is the instanton approach, allowing us to avoid difficulties associated with the replica method. The Laplace transform of the function (2) can be presented as the functional integral,[8]

$$\Pi(\lambda)=\langle e^{\lambda[\varepsilon(0,\zeta_1)-\varepsilon(0,\zeta_2)]}\rangle=\int D\varepsilon D\psi e^{S[\varepsilon,\psi]}, \tag{4}$$

with the effective action

$$\begin{aligned}S[\varepsilon,\psi]=i\int dx d\zeta \psi(x,\zeta)\left[\frac{\partial\varepsilon}{\partial x}+\frac{1}{2\kappa}\left(\frac{\partial\varepsilon}{\partial\zeta}\right)^2-\frac{T}{2\kappa}\frac{\partial^2\varepsilon}{\partial\zeta^2}\right]\\ -\beta\int dx d\zeta d\zeta'\,\psi(x,\zeta)K(\zeta-\zeta')\psi(x,\zeta')\\ +\lambda[\varepsilon(0,\zeta_1)-\varepsilon(0,\zeta_2)].\end{aligned} \tag{5}$$

To arrive at Eq. (5), we have used the identity

$$\int D\varepsilon\,\delta\left[\frac{\partial\varepsilon}{\partial x}+\frac{1}{2\kappa}\left(\frac{\partial\varepsilon}{\partial\zeta}\right)^2-\frac{T}{2\kappa}\frac{\partial^2\varepsilon}{\partial\zeta^2}-V\right]=\text{const},$$

which is a consequence of causality, and subsequently performed averaging over the random potential V.

The field theory (5) is essentially two dimensional. To move further, we extend the technique developed by Gurarie and Migdal[11] (GM), who studied velocity correlations in the Burgers equation. That is, in order to describe the tails of the correlation function (2), we have to find the instanton (saddle-point) trajectory and calculate the action S_{in} at this trajectory. The result for $\Pi(\lambda)$ with the exponential accuracy reads

$$\Pi(\lambda)=\exp[S_{in}(\lambda)-S_{in}(0)]. \tag{6}$$

Prefactors can be obtained systematically by expanding the action around the instanton path, but this goes beyond the scope of this paper.

The saddle-point equations for the effective action $S[\varepsilon,\psi]$ have the form

$$\frac{\partial\varepsilon}{\partial x}+\frac{1}{2\kappa}\left(\frac{\partial\varepsilon}{\partial\zeta}\right)^2-\frac{T}{2\kappa}\frac{\partial^2\epsilon}{\partial\zeta^2}=-2i\beta\int d\zeta' K(\zeta-\zeta')\psi(x,\zeta'), \tag{7}$$

$$\begin{aligned}\frac{\partial\psi}{\partial x}+\frac{1}{\kappa}\frac{\partial}{\partial\zeta}\left(\psi\frac{\partial\varepsilon}{\partial\zeta}\right)+\frac{T}{2\kappa}\frac{\partial^2\psi}{\partial\zeta^2}\\ =-i\lambda\,\delta(x)[\delta(\zeta-\zeta_1)-\delta(\zeta-\zeta_2)].\end{aligned} \tag{8}$$

II. LONG-RANGE DISORDER

We assume first that the function K is only slightly changed on the scale of $\zeta_1-\zeta_2$. It can be then expanded, $K(y)=k_0-k_1y^2/2$. In the subsequent analysis, we follow GM.

Note that Eq. (8) only has nonzero solutions with finite action for $x<0$ (''diffusion in reverse time''). The field ψ is discontinuous at $x=0$; integrating Eq. (8) between -0 and $+0$, we obtain the boundary condition

$$\psi(x=-0,\zeta)=i\lambda[\delta(\zeta-\zeta_1)-\delta(\zeta-\zeta_2)]. \tag{9}$$

To solve the saddle-point equations, we disregard the term with $\partial^2\psi/\partial\zeta^2$ in Eq. (8) for a while. Then, due to Eq. (9), the function ψ always remains the sum of two δ functions,

$$\tilde{\psi}(x,\zeta)=i\mu(x)[\delta(\zeta-\rho_1(x))-\delta(\zeta-\rho_2(x))], \tag{10}$$

where the tilde means that this solves the ''incomplete'' equation. Equation (9) implies $\mu(0)=\lambda$, $\rho_{1,2}(0)=\zeta_{1,2}$. From Eq. (7) we then find $\varepsilon(x,\zeta)=\tilde{a}(x)+\tilde{b}(x)\zeta$, with

$$\tilde{a}'+\tilde{b}^2/2\kappa=-\beta k_1\mu(\rho_1^2-\rho_2^2),$$

$$\tilde{b}'=2\beta k_1\mu(\rho_1-\rho_2).$$

Substituting this back into Eq. (8), we find $\mu(x)=\lambda$ and $\rho_1(x)-\rho_2(x)=\zeta_1-\zeta_2$. These solutions and, consequently, the resulting form of the distribution function, are very different from those for Burgers equation.[11]

Now we include the term with $\partial^2\psi/\partial\zeta^2$ into consideration. We look for a solution of Eq. (7) in the form of a linear function of ζ, $\varepsilon=a(x)+b(x)\zeta$ (the consistency is checked afterwards). The *linear* equation (8) is easily solved, and yields

$$\psi(x,\zeta)=\frac{i\lambda\kappa^{1/2}}{(2\pi T|x|)^{1/2}}[e^{[\kappa(\zeta-\rho_1)/2Tx]}-e^{[\kappa(\zeta-\rho_2)/2Tx]}], \tag{11}$$

with $\rho_1'=\rho_2'=b(x)/\kappa$. Substituting this into the right-hand side of Eq. (7), we find that the latter is temperature independent, i.e., it remains the same as that without the diffusion term. Thus, the linear ansatz for ε is consistent. Also, we find $b'=2\beta k_1\lambda(\rho_1-\rho_2)$. The solution is then $\rho_{1,2}(x)=\zeta_{1,2}+b_0x/\kappa+\beta k_1\lambda(\zeta_1-\zeta_2)x^2/\kappa$.

Next, we calculate the action S along the instanton trajectory. As seen from the saddle-point equations, the first term in Eq. (5) is just double of the second one. Multiplying Eq. (8) by ε, integrating it over x (from $-\infty$ to -0) and over ζ, and comparing with Eq. (7), we find

$$\lambda[\varepsilon(0,\zeta_1)-\varepsilon(0,\zeta_2)]=-2\beta\int\psi K\psi-\frac{i}{2\kappa}\int dx d\zeta\,\psi\left(\frac{\partial\varepsilon}{\partial\zeta}\right)^2.$$

Since ε linearly depends on ζ, and $\int\psi d\zeta=0$, the last term in the right-hand side vanishes. The instanton action acquires the form

$$S_{in}=-\beta\int_{-\infty}^{-0}dx\int d\zeta d\zeta'\,\psi(x,\zeta)K(\zeta-\zeta')\psi(x,\zeta'). \tag{12}$$

Substituting the instanton solution (11) into Eq. (12), we find that the term proportional to k_0 in the effective action vanishes, while the contribution with k_1 diverges in the limit of large negative x. This is because our consideration is limited to distances shorter than the correlation length of the random

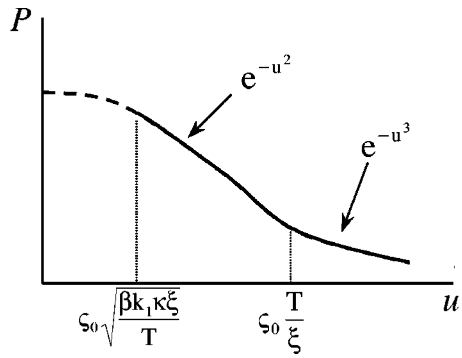

FIG. 1. Distribution function $P(u)$ for long-ranged correlated disorder, Eq. (14).

potential $\xi\sim(k_0/k_1)^{1/2}$, so that we could replace $K(y)$ by its expansion. This cutoff effectively defines the constant b_0. Replacing integrals over ζ in Eq. (12) in infinite limits by integrals from $-\xi/2$ to $\xi/2$ and calculating them, we arrive at the cutoff for x,

$$x_c=\begin{cases}-\kappa\xi^2/T, & T>T_c\\ -(\kappa\xi/\beta k_1\lambda\zeta_0)^{1/2}, & T<T_c,\end{cases} \tag{13}$$

with $\zeta_0=\zeta_1-\zeta_2>0$ and $T_c=(\beta k_1\kappa\zeta_0\xi\lambda)^{1/2}$. The instanton action reads $S_{in}=\beta k_1\zeta_0^2\lambda^2|x_c|$. Note that, since our cutoff procedure is somewhat arbitrary, we have removed all numerical factors from the action. Laplace transforming Eq. (6), we arrive at the expression

$$P(u)=\begin{cases}e^{-c'Tu^2/(\beta k_1\kappa\xi^2\zeta_0^2)}, & u\ll T\zeta_0/\xi\\ e^{-cu^3/(\beta k_1\kappa\xi\zeta_0^3)}, & u\gg T\zeta_0/\xi,\end{cases} \tag{14}$$

which is illustrated on Fig. 1, with $c\sim c'\sim 1$.

The result (14) was obtained by instanton approach, which means it has to be exponentially small. In particular, this requires $u>0$. We thus get the conditions $u\gg\zeta_0(\beta k_1\kappa\xi)^{1/3}$ for the elasticity-controlled regime $[P(u)\propto\exp(-u^3)]$ and $\zeta_0(\beta k_1\kappa\xi^2/T)^{1/2}\ll u\ll\zeta_0T/\xi$ for the temperature-controlled regime $[P(u)\propto\exp(-u^2)]$. The latter result only holds for temperatures higher than $T_d=\xi^{4/3}(\beta k_1\kappa)^{1/3}$, which does not depend on ζ_0 and has a meaning of depinning temperature—the typical pinning energy on the Larkin correlation length. Above the depinning temperature correlation functions decay faster than exponentially.[1] In particular, note the similarity between the lower line of Eq. (14) and the corresponding expression for the distribution function of the full energy of the string in Ref. 10.

III. SHORT-RANGED DISORDER

We take now $K(y)=\delta(y)$. At $x=0$, the field ψ is a set of a δ peak (located at $\zeta=\zeta_1$) and a δ dip ($\zeta=\zeta_2$), see Eq. (9). As we trace the evolution in reverse time, these features smear and move. For short times, before (if ever) they intersect or smear so much that they become indistinguishable, ψ remains a peak-dip function, centered at $\zeta=\rho_1(x)$ and $\zeta=\rho_2(x)$, and sharply vanishing away from these points.

Let us calculate the energy created by a peak-shaped field ψ, located at $\zeta=\rho$, far from this point. At large distances from the peak, the solution must have a scaling form. Equation (7) with the zero right-hand side allows for only one scaling solution $\varepsilon=f((\zeta-\rho)/x^{1/2})$ vanishing at $x\to-\infty$, which at large scales, where the term with the second derivative can be disregarded, becomes $\varepsilon=\kappa(\zeta-\rho)^2/2x$. For $|\zeta-\rho|<(Tx/\kappa)^{1/2}$, the diffusion term is important, and this simple form of the scaling solution does not apply.

Our solution has an obvious drawback: It diverges as x goes to zero. To amend this, we write

$$\varepsilon=\frac{\kappa(\zeta-\rho)^2}{2(x-x_0)},\quad x_0>0. \tag{15}$$

The regularization constant x_0 plays an important role and is found from the following considerations. For $x=0$, the energy given by Eq. (15) is $\varepsilon_0=-\kappa\zeta^2/x_0$. In the scaling regime, the only one relevant length scale is $\zeta_0=\zeta_1-\zeta_2$, and the only relevant energy scale is temperature. Therefore, $T\sim\kappa\zeta_0^2/x_0$, whence $x_0\sim\kappa\zeta_0^2/T$. The numerical factor remains undetermined, but can be deduced from numerical solutions of Eqs. (7) and (8).

Now we return to Eq. (8) and solve it at small $|x|$, when the peak and the dip are well separated. We write $\psi(x,\zeta)$ in the form $\psi_{peak}+\psi_{dip}$. The peak [located near $\zeta=\rho_1(x)$] then experiences the energy produced by itself and the energy produced by the dip. The energy of an isolated peak (15) grows as ζ^2 far from the peak, and hence at the point ρ_1, the energy produced by the dip is much greater than the peak contribution. If, in addition, $\rho_2-\rho_1\gg(T(x_0-x)/\kappa)^{1/2}$, we can use Eq. (15) to evaluate the coefficient $\partial\varepsilon/\partial\zeta$ in Eq. (8). We get then the equation

$$\frac{\partial\psi_{peak}}{\partial x}+\frac{\psi_{peak}}{x-x_0}+\frac{\rho_1-\rho_2}{x-x_0}\frac{\partial\psi_{peak}}{\partial\zeta}+\frac{T}{2\kappa}\frac{\partial^2\phi}{\partial\zeta^2}=0, \tag{16}$$

with the boundary condition $\psi_{peak}(x=0)=i\lambda\delta(\zeta-\zeta_1)$.

Writing a similar equation for the dip ($\zeta\approx\rho_2$) and solving both equations, we obtain

$$\begin{Bmatrix}\psi_{peak}\\ \psi_{dip}\end{Bmatrix}=\pm\frac{i\lambda x_0\kappa^{1/2}}{(2\pi T|x|)^{1/2}}\frac{1}{x_0-x}\exp\left(-\frac{\kappa(\zeta-\rho_{1,2})^2}{2T|x|}\right),$$

$$\begin{Bmatrix}\rho_1'\\ \rho_2'\end{Bmatrix}=\pm\frac{\rho_1-\rho_2}{x-x_0}. \tag{17}$$

As follows from Eq. (17), the positions of the peak and the dip are pulled apart. The distance between them grows as $(x-x_0)^2$; at the same time, they smear as $|x|^{1/2}$; thus, they never overlap and can be considered as well separated at any x. Using Eq. (17) to calculate the instanton action, we get

$$S_{in}=\tilde{a}\beta\lambda^2\kappa\zeta_0/T,\quad \zeta_0=\zeta_1-\zeta_2,\quad \tilde{a}\sim 1, \tag{18}$$

which translates into the expression for the distribution function

$$P(u)=\exp\left(-\frac{aTu^2}{\beta\kappa\zeta_0}\right),\quad a\sim 1. \tag{19}$$

The expression (19) is valid for temperatures much higher than $\beta\kappa\zeta_0/u^2$ and is an analog of the upper line of Eq. (14). We conjecture that the long-u tail of the distribution function [the analog of the lower line of Eq. (14)] also exists, though we were not able to obtain it with this method of solving Eqs. (7) and (8).

IV. DISCUSSION AND CONCLUSIONS

Equations (14) and (19) constitute the central result of this paper. The function $P(u)$ has a meaning of the probability that the difference of energies of a string of the same length and the transverse displacements ζ_1 and ζ_2, equals u. An instanton solution corresponds to the situation where at ζ_1 there is a minimum of the potential energy while at ζ_2 the *neighboring* maximum exists. $P[u]$, thus, measures the height of the barrier of the free-energy relief. For large u, to find a high barrier is quite improbable, consequently the result is exponentially small.

If the string has a finite transverse size ξ_0, the correlation radius ξ of the disordered potential is of order of ξ_0. Equation (14) thus describes the case when the string is displaced at a distance small compared with its transverse dimension, and Eq. (19) applies in the opposite regime of long distance.

These results have an important conceptual value, since they describe correlation functions of energy at different positions for the directed polymer problem. We argue now that they also provide certain predictions for experimentally observable dynamical properties of the string. We suggest two types of experiments, which certainly do not exhaust all the opportunities.

(i) Creep of domain walls. If a domain wall moves from one position to another one, the distance between the two positions can be measured. The quantity ζ_0, which is the distance between a position and a maximum separating the two, is not known, but can be determined if the wall travels back and forth. The tunneling rate is determined by the height of the barrier u. A similar information can be extracted from the random telegraph noise resulting from the motion of the string between two positions. The time spent in each position is determined by the barrier to be overcome seen from this position. These considerations do not take into account a possibility of macroscopic quantum tunneling, which has to be considered separately.

(ii) Distribution of pinning energies. The string is (locally) depinned if it moves from one free-energy minimum to the next one. The pinning energy corresponds precisely to the quantity u above. However, globally one has to average over the distribution of the displacements ζ_0, i.e., to know the distribution of distances between adjacent minima and maxima of the random potential. Measurements of the distributions of the pinning energy compared with our results can provide an information about this distribution, which, to our knowledge, has not been previously discussed.

In conclusion, we have developed an instanton approach to calculations of various correlation functions describing statistical behavior of the elastic string in the two-dimensional disordered potential. We applied our technique to the investigation of correlations of free energies corresponding to different low-lying metastable positions of the string. We have found the asymptotic behavior of such energy-energy correlations for the moderate spatial scales (within Larkin correlation length) and the large scales, exceeding Larkin length. The latter situation corresponds to the conditions of the creep dynamics. We have discussed applications of our results to the dynamic response and noise in various two-dimensional systems such as domain walls, vortices, and dislocations in thin films.

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