Thermodynamics of the Superfluid Dilute Bose Gas with Disorder

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We generalize the Beliaev-Popov diagrammatic technique for the problem of interacting dilute Bose gas with weak disorder. Averaging over disorder is implemented by the replica method. The low-energy asymptotic form of the Green function confirms that the low-energy excitations of the superfluid dirty-boson system are sound waves with velocity renormalized by the disorder and additional dissipation due to the impurity scattering. We find the thermodynamic potential and the superfluid density at any temperature below the superfluid transition temperature (but outside the Ginzburg region) and derive the phase diagram in temperature vs disorder plane.

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Superfluidity in random environments enjoys a long-standing yet intense attention. The effect of disorder on the behavior of systems possessing long-range correlations is central to contemporary condensed matter physics, and superfluid Bose gas offers an exemplarily unique and accessible tool for both experimental and theoretical researches. One of the fascinating properties of such systems is their ability to maintain superfluidity (i.e., long-range correlations) even in the strongly disordered environment. He^4, for example, remains superfluid when absorbed in porous media [1]. The problem of influence of disorder on superfluidity (and on its close analog superconductivity) has been under extensive theoretical attack (see seminal works [2,3]) and remarkable progress in qualitative understanding of disordered Bose systems was achieved. Recent papers [4,5] discussed a continuum model of the dilute interacting Bose gas in a random potential. The advantages of this model are that (i) it is microscopically related to the original problem and (ii) it is very well understood in the clean limit. The proposed model describes, in particular, the quasiparticle dissipation and depletion of superfluidity at zero temperature and marked an important step towards a quantitative description of disordered Bose systems.

In this Letter, building on the model of Refs. [4,5], we develop a systematic diagrammatic perturbation theory for the dilute Bose gas with weak disorder at finite temperatures below the superfluid transition temperature T_s. We obtain disorder corrections to the thermodynamic potential which completely determine thermodynamic properties of the superfluid system. We derive for the first time the disorder-induced shift of T_s resulting from disorder scattering of quasiparticles with energy $\epsilon \sim T$. We find that the superfluid density decreases monotonically with the temperature. This agrees with the experimental data, while being in some contradiction with the theoretical result of Ref. [4] where a nonmonotonic temperature dependence of superfluid density was claimed. In the limit $T \to 0$ our theory reproduces all the results of Refs. [4,5].

The model.—The starting point of our model is the Lagrangian density:

$$\mathcal{L} = -\varphi^*[-\nabla_r^2/2m - \mu + u(r) + \partial_\tau]\varphi - g\varphi^*\varphi^*\varphi\varphi\,,$$

where $\varphi = \varphi(r,\tau)$ is the field representing Bose particles, r is the real space coordinate, τ is the Matsubara time, and $u(r)$ is the disorder potential. As usual we consider a soft interaction potential $g(r)$ and use the Born approximation $g = 4\pi\lambda/m$ to relate the interaction constant $g = \int g(r)\,d^d r$ to the scattering length λ [6]. Taking Gaussian δ-correlated disorder $\langle u(r_1)u(r_2)\rangle = \kappa\delta(r_1 - r_2)$ we derive the effective replicated action in a form:

$$S = -\sum_p \varphi_\alpha^*(p)\,(k^2/2m - \mu - i\omega)\varphi_\alpha(p) + V_i + V_d\,,$$

with $p = (k,\omega)$ and the interaction parts V_i and V_d,

$$V_i = -\frac{g}{2\beta V}\sum_{k,\omega,\alpha}\varphi_\alpha^*(k_1,\omega_1)\varphi_\alpha^*(k_2,\omega_2) \times \varphi_\alpha(k_3,\omega_3)\varphi_\alpha(k_4,\omega_4)\,,$$

$$V_d = \frac{\kappa}{2V}\sum_{k,\omega,\alpha,\beta}\varphi_\alpha^*(k_1,\omega_1)\varphi_\alpha(k_3,\omega_1) \times \varphi_\beta^*(k_2,\omega_2)\varphi_\beta(k_4,\omega_2)\,,$$

where $\alpha,\beta = 1,\ldots,n$ are the replica indexes, and the conservation of total momentum $k_1 + k_2 = k_3 + k_4$ (in V_i and V_d) and of total "energy" $\omega_1 + \omega_2 = \omega_3 + \omega_4$ (in V_i) is assumed. The corresponding vertices are presented in Fig. 1a. Below T_s we separate the condensate contribution by shifting the fields $\varphi_\alpha \to a\sqrt{\beta V}\,\delta_{k,0}\delta_{\omega,0} + \varphi'_\alpha$ and define the Green functions of the fields φ'_α by

$$G_{\alpha\beta}(p) = \langle\varphi'_\alpha(p)\varphi'^*_\beta(p)\rangle, \quad F_{\alpha\beta}(p) = \langle\varphi'_\alpha(p)\varphi'_\beta(-p)\rangle. \tag{1}$$

Defining also the functions $\bar{G}_{\alpha\beta}(p) = \langle\varphi'^*_\alpha(-p) \times \varphi'_\beta(-p)\rangle$, $\bar{F}_{\alpha\beta} = \langle\varphi'^*_\alpha(-p)\varphi'^*_\beta(p)\rangle$ we introduce the matrix Green function and the matrix self-energy

$$\mathcal{G}_{\alpha,\beta} = \begin{bmatrix} G_{\alpha\beta} & F_{\alpha\beta} \\ \bar{F}_{\alpha\beta} & \bar{G}_{\alpha\beta}\end{bmatrix}, \qquad \Sigma_{\alpha\beta} = \begin{bmatrix} A_{\alpha\beta} & B_{\alpha\beta} \\ \bar{B}_{\alpha\beta} & \bar{A}_{\alpha\beta}\end{bmatrix}, \tag{2}$$

that are related by the Dyson equation

$$\mathcal{G}^{-1} = (p^2/2m - \mu - i\omega\tau_3)\delta_{\alpha\beta} + \Sigma\,, \tag{3}$$

where τ_3 is the Pauli matrix. The condensate density is uniformly distributed in the replica space; therefore, Green

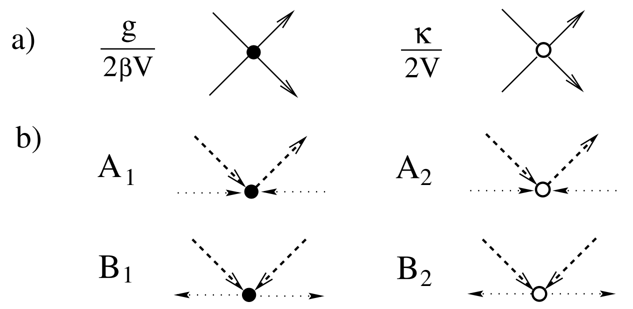

FIG. 1. Interaction and disorder vertices (a) and first order contributions to self-energies (b). The dashed lines in (b) represent the condensate.

function $\mathcal{G}$ and self-energy Σ can be presented as

$$\mathcal{G}_{\alpha\beta} = \mathcal{G}_1\delta_{\alpha\beta} + \mathcal{G}_2 R_{\alpha\beta}\,, \qquad \Sigma_{\alpha\beta} = \Sigma_1\delta_{\alpha\beta} + \Sigma_2 R_{\alpha\beta}\,, \tag{4}$$

where $R_{\alpha\beta}$ is the matrix with all elements equal to 1. From the $n \to 0$ limit of Eq. (3) we obtain that the replica diagonal part of the Green function is determined by the replica diagonal part of the self-energy

$$\mathcal{G}_1^{-1} = (p^2/2m - \mu - i\omega\tau_3) + \Sigma_1\,, \qquad \mathcal{G}_2 = -\mathcal{G}_1\Sigma_2\mathcal{G}_1\,, \tag{5}$$

and the poles of the Green function $\mathcal{G}$ are determined by the poles of the Green function G_1. Because of the Goldstone theorem, below the condensation temperature the function G_1 has a pole at $p = 0$, and from (5) we find

$$A_1(0) - B_1(0) = \mu\,. \tag{6}$$

Our diagrammatic technique parallels that by Beliaev, the difference being that we include the disorder vertex V_d along with the interaction vertex V_i. The corresponding diagrams are shown in Fig. 1b.

$$A_1 = 2ga^2, \qquad B_1 = ga^2, \qquad A_2 = B_2 = -\kappa\beta a^2. \tag{7}$$

From Eq. (6) we obtain $ga^2 = \mu$ and using Eq. (5) for the Green functions we have

$$G_1(p) = \frac{k^2/2m + \mu + i\omega}{\varepsilon^2(k) + \omega^2}\,, \qquad F_1(p) = \frac{-\mu}{\varepsilon^2(k) + \omega^2}\,, \tag{8}$$

$$G_2(p) = F_2(p) = \frac{\kappa\beta a^2}{(k^2/2m + 2\mu)^2}\,\delta_{\omega,0}\,, \tag{9}$$

where $\varepsilon(k) = \sqrt{(k^2/2m)^2 + \mu k^2/m}$. We see that the spectrum of quasiparticles is not affected by disorder in the leading order. The Bose gas density is given by

$$n = a^2 + n_1 + n_2 \tag{10}$$

$$n_1 = \frac{T}{V}\sum_p G_1(p), \qquad n_2 = \frac{T}{V}\sum_{k,\omega=0} G_2(k)\,. \tag{11}$$

Zero temperature.—At zero temperature in three dimensions using Eqs. (8) and (9) we obtain

$$n_1 = \frac{8}{3\sqrt{\pi}}(\lambda n)^{3/2}, \qquad n_2 = \frac{\kappa}{4\pi}\,\frac{a^2 m^{3/2}}{\sqrt{\mu}}\,. \tag{12}$$

The contribution n_1 represents the quasiparticle density due to quantum fluctuations; in the leading order it coincides with the well known answer for the pure case. The contribution n_2 represents the density of the nonuniform part of the condensate. For the theory to be applicable both n_1 and n_2 should be much less than the total density

$$\lambda' = \lambda n^{1/3} \ll 1, \qquad \kappa' = \kappa m^2/(8\pi^{3/2}\sqrt{\lambda n}) \ll 1\,. \tag{13}$$

The first relation is the usual low-density condition; the second one ensures that the uniform part of the condensate is not strongly affected by disorder. To relate the condensate density with the chemical potential we need to improve the leading order result $ga^2 = \mu$ considering next order corrections to (6). The second order corrections $A_1^{(2)}, B_1^{(2)}$ contain the contributions $A_1^{(2,i)}, B_1^{(2,i)}$ from the quasiparticle interactions and the disorder contributions $A_1^{(2,d)}, B_1^{(2,d)}$ linear in κ. Corrections $A_1^{(2,i)}, B_1^{(2,i)}$ presented in Fig. 2a exactly coincide with ones studied in [7] for the pure Bose gas. The disorder corrections $A_1^{(2,d)}, B_1^{(2,d)}$ presented in Fig. 2b contain all the diagrams that are (i) linear in disorder coupling κ and (ii) have a similar structure with ones shown in Fig. 2a. Here we present the answers for their linear combinations $\Sigma_\pm^{(2,d)} = A_1^{(2,d)} \pm B_1^{(2,d)}$:

$$\mathrm{Re}\Sigma_-^{(2,d)}(q,\omega) = gn_2 - \frac{\kappa}{V}\sum_k \frac{k^4 G'_-(k+q,\omega)}{(k^2+4\mu m)^2}\,,$$
$$\mathrm{Re}\Sigma_+^{(2,d)}(q,\omega) = 3gn_2 - \frac{\kappa}{V}\sum_k \frac{(k^2-8\mu m)^2 G'_+(k+q,\omega)}{(k^2+4\mu m)^2}\,, \tag{14}$$

where $G'_\pm(p) = \mathrm{Re}[G_1(p) \pm F_1(p)]$. The self-energy B

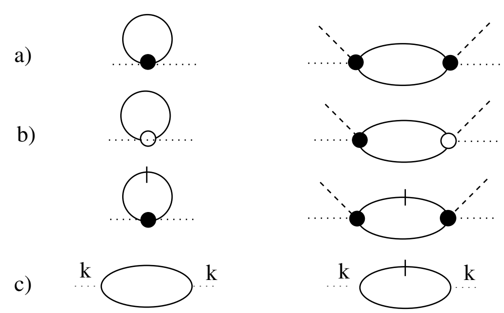

FIG. 2. Second order contributions to the self-energies A_1, B_1 due to interaction (a), and due to disorder (b). The solid lines represent the Green functions G_1, F_1; the crossed solid lines represent the Green functions G_2, F_2. The diagrams in part (c) represent two contributions to the normal density $n_n^{(1)}$ and $n_n^{(2)}$.

is real while A is complex with

$$\mathrm{Im}A^{(2,d)}(q,\omega) = \sum_k \frac{\kappa\omega k^2/V}{\varepsilon^2(k+q)+\omega^2}\,\frac{8\mu m - k^2}{(k^2+4\mu m)^2}\,. \tag{15}$$

Using Eq. (14) along with the well known result [7] for $\Sigma_{-}^{(2,i)}$ from Eq. (6) we obtain

$$\mu = ga^2\left(1+\frac{40}{3}\sqrt{\frac{n\lambda^3}{\pi}}\right)+gn_2 - \frac{\kappa}{V}\sum_k \frac{2m\,k^2}{(k^2+4\mu m)^2}\,.$$

Combining this equation with Eq. (10) we obtain the relation between the density and chemical potential

$$\mu = gn\left(1+\frac{32}{3}\sqrt{\frac{n\lambda^3}{\pi}}\right) - \frac{\kappa}{V}\sum_k \frac{2m\,k^2}{(k^2+4\mu m)^2}\,. \tag{16}$$

Using the thermodynamic relation $\mu = dE/dN$ along with the leading order relation $gn = \mu$ we obtain the energy

$$\frac{E}{V} = \frac{gn^2}{2}\left(1+\frac{128}{15}\sqrt{\frac{\lambda^3 n}{\pi}}\right) - \frac{\kappa}{V}\sum_k \frac{2n\,m}{k^2+4\mu m}\,, \tag{17}$$

which agrees with [5]. The integrals over k in Eqs. (16) and (17) are ultraviolet divergent; it is the consequence of the white noise assumption for the disorder correlation function. This divergence is not relevant for the low-energy physics since it could be absorbed in the renormalization of energy and chemical potential: $E \to E + n\kappa\sum_k 2m/k^2$, $\mu \to \mu + \kappa\sum_k 2m/Vk^2$.

The superfluid density n_s can be found from the normal density n_n which is determined by the transverse current-current correlator $n_s = n - n_n$. In the leading order n_n is given by the diagrams shown in Fig. 2c:

$$n_n = n_n^{(1)} + n_n^{(2)}, \tag{18}$$

where $n_n^{(1)}$ is the normal density of the clean system $n_n^{(1)} = \frac{T}{3mV}\sum_p k^2[G_1^2(p) - F_1^2(p)]$ which after summation over Matsubara frequencies may be written as

$$n_n^{(1)} = \frac{1}{12mTV}\sum_k \frac{k^2}{\sinh^2[\epsilon(k)/2T]}\,, \tag{19}$$

and $n_n^{(2)}$ is the disorder correction,

$$n_n^{(2)} = \frac{T}{3mV}\sum_p 2k^2[G_1(p) - F_1(p)]G_2(p) = \frac{4}{3}n_2\,. \tag{20}$$

Thus at zero temperature $n_n^{(1)} = 0$ and superfluid density becomes $n_s = n - 4n_2/3$ in agreement with Refs. [4,5].

The second order corrections to the self-energies (14) and (15) can be calculated explicitly for small q, ω leading to the following low-energy retarded Green function:

$$G_1^R(k,\omega) = \frac{m}{n_s}\,\frac{c^2a^2}{c^2k^2 - \omega^2 - 2ick\Gamma(k)}\,, \qquad \omega > 0, \tag{21}$$

where the sound velocity c is related to the sound velocity of the clean system c_0 by $c^2 = c_0^2(1 + 5n_2/3n)$, and $\Gamma(q) = \kappa q^4/24m^2c^3\pi$ is the dissipation of quasiparticles due to disorder scattering. The dissipation due to quasiparticle (phonon) scattering is known to be of a higher power of q: $\Gamma^{(ph)}(q) \sim q^5$. The results for sound velocity c and quasiparticle dissipation $\Gamma(q)$ are in agreement with [5].

Bose condensation temperature.—Now we turn to finite temperatures. At temperatures above the condensation temperature T_c the self-energy is given by the first diagrams of Figs. 2a and 2b.

$$A_1 = 2gn - \frac{\kappa}{V}\sum_k G(k,\omega)\,, \tag{22}$$

and $A_2, B_1, B_2 = 0$. Taking the sum over k in (22) we obtain the Green function

$$G^{-1}(p) = \frac{k^2}{2m} - \tilde{\mu} - i\omega + \kappa\frac{(2m)^{3/2}}{4\pi}\sqrt{|\tilde{\mu}| - i\omega}\,, \tag{23}$$

where the $\tilde{\mu} = \mu - 2gn + \kappa\sum_k 2m/Vk^2$. Inserting the Green function (23) into Eq. (11) in the leading order in κ at the condensation temperature ($\tilde{\mu} = 0$) we get

$$n = \zeta_{3/2}(mT_c/2\pi)^{3/2} + \kappa T_c m^3/4\pi^2. \tag{24}$$

Solving this equation for T_c we find the shift of the condensation temperature due to disorder:

$$T_c = T_c^{(0)}(1 - \kappa T_c^{(0)} m^3/6\pi^2 n)\,, \tag{25}$$

where $T_c^{(0)} = 2\pi(n/\zeta_{3/2})^{2/3}/m$ is the Bose condensation temperature of the ideal gas with $\zeta_{3/2} \approx 2.612$. At finite g only a microscopic amount of particles may condense into a local potential well [8]; this effect leads to the smearing of the condensation transition making T_c to be the crossover temperature between the normal phase and a phase where bosons are locally condensed. The true phase transition takes place when the chemical potential reaches the mobility edge [2]; it may also be obtained from the condition $n_s = 0$ (see below).

Thermodynamics at $T \sim T_s$.—The self-energies at $T \sim T_s$ are still given by the diagrams presented in Figs. 1 and 2, but the first diagram of Fig. 1a should be included already in the first order approximation since the density of quasiparticle excitations n_1 at $T \sim T_s$ is of the order of total density. This diagram results only in the shift of the chemical potential $\mu \to \mu - 2gn_1 = \bar{\mu}$, and the Green functions are still given by Eqs. (8) and (9) but with $\mu \to \bar{\mu}$. The density in the leading approximation is still given by Eqs. (10) and (11), but now n_1 is not a small correction and should therefore be calculated with a higher accuracy. The main contribution to n_1 comes from the energies $\epsilon \sim T \gg \bar{\mu}$. At these energies the Green function is given by

Eq. (23) and, thus, the disorder correction to n_1 is the same as in Eq. (24), i.e., $\kappa T m^3/4\pi^2$. Taking into account this correction and using Eq. (11) we obtain

$$n_1 = nT'^{3/2} - \frac{\bar{\mu}^{1/2} m^{3/2} T}{2\pi} + \frac{\kappa m^3 T}{4\pi^2}, \qquad n_2 = \frac{\kappa a^2 m^{3/2}}{4\pi\sqrt{\bar{\mu}}}, \tag{26}$$

where $T' = T/T_c^{(0)}$. To relate the chemical potential with the condensate density we consider next order corrections to the leading order result $ga^2 = \bar{\mu}$ following from Eq. (6). Using $\Sigma_-^{(2,i)}$ from [9] and Eq. (14) we obtain

$$\frac{\bar{\mu}}{g} = n_2 + a^2 - \sum_k \frac{(\kappa/g) 2mk^2}{(k^2 + 4\bar{\mu}m)^2} - \frac{3\bar{\mu}^{1/2} m^{3/2} T}{2\pi} + \frac{\kappa m^3 T}{2\pi^2},$$

and combining this equation with Eq. (26) we get an equation relating density and chemical potential,

$$n = \mu/g - nT'^{3/2} + \bar{\mu}^{1/2} m^{3/2} T/\pi + \Delta n^{(d)}, \tag{27}$$

where $\Delta n^{(d)}$ is the disorder contribution

$$\Delta n^{(d)} = \frac{\kappa}{gV} \sum_k \frac{2mk^2}{(k^2 + 4\bar{\mu}m)^2} - \frac{\kappa m^3 T}{4\pi^2}. \tag{28}$$

Using the relation $N = -d\Omega/d\mu$ we eventually obtain the disorder correction to the thermodynamic potential:

$$\frac{\delta\Omega^{(d)}}{V} = -\frac{\kappa}{Vg} \sum_k \frac{2\bar{\mu}m}{k^2 + 4\bar{\mu}m} + \frac{\kappa m^3 T \mu}{4\pi^2}. \tag{29}$$

Superfluid density at $T \sim T_s$.—The disorder contribution to the normal density $n_n^{(2)}$ at $T \sim T_s$ is related to n_2 through Eq. (20) with n_2 defined by Eq. (26) that takes into account the chemical potential shift $\mu \to \bar{\mu}$. Deriving the contribution $n_n^{(1)}$ one needs to consider that according to Eq. (23) the spectrum of quasiparticles is affected by the disorder at energies $\epsilon \sim T_s$, which results in

$$n_n^{(1)} = n_n^{(cl)} + \kappa m^3 T/4\pi^2, \tag{30}$$

where $n_n^{(cl)}$ is the normal density of the clean system defined by Eq. (19) with the spectrum $\epsilon^2(k) = (k^2/2m)^2 + \bar{\mu}k^2/m$. Introducing dimensionless condensate density $a' = \sqrt{1 - T'^{3/2}}$ we write the superfluid density as

$$n_s/n = n_s^{(cl)}/n - 4\kappa' a'/3 - 4\kappa'\sqrt{\pi\lambda'}\, T'/\zeta_{3/2}^{2/3}, \tag{31}$$

where $n_s^{(cl)}$ is the superfluid density of the clean system $n_s^{(cl)} = n - n_n^{(cl)}$. The dependence of n_s on temperature for different amounts of disorder is presented in Fig. 3. Taking $n_s = 0$ in Eq. (31) for the superfluid transition temperature at $\kappa' \gg \sqrt{\lambda'}$ we have

$$T_s = T_c^{(0)}(1 - 32\kappa'^2/27). \tag{32}$$

At $\kappa' \lesssim \sqrt{\lambda'}$ T_s coincides with condensation crossover line (25) but this region is already at the boundary of the Ginzburg region $\delta T' \sim \lambda'$ where nonperturbative effect of interactions become important [10].

In conclusion, we have developed a regular diagrammatic approach that enables a quantitative description of thermodynamics of superfluid dilute Bose gas in random environment at finite temperatures. We have found disorder corrections to condensation temperature, thermodynamic potential, and the superfluid density. Our results agree favorably with the experimental findings.

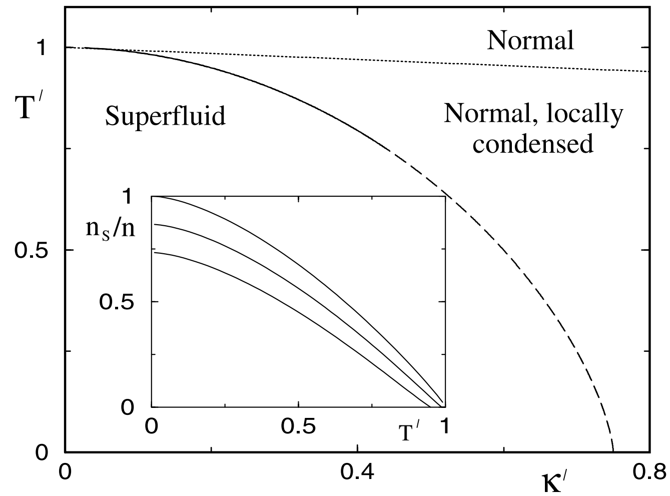

FIG. 3. The temperature-disorder phase diagram resulting from Eqs. (25) and (31) for $\lambda' = 0.03^2$. The dashed part of the boundary of the superfluid phase corresponding to the region $\kappa \sim 1$ should be understood as an extrapolation. The inset shows the superfluid density n_s/n dependence on temperature $T' = T/T_c^{(0)}$ for different amounts of disorder: $\kappa' = 0, 0.1, 0.2$ (top to bottom). The dotted line represents the condensation crossover temperature determined by Eq. (25).

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