Surface Criticality and Multifractality in the Spin Quantum Hall System

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Abstract

We examine the surface of a two-dimensional system exhibiting the spin quantum Hall transition. A supersymmetric (SUSY) formalism enables us to map our surface problem to a classical percolation model, similar to the scheme of Gruzberg, Ludwig and Read [Phys. Rev. Lett. 82, 4524 (1999)], with reflecting boundary conditions but with the full intact supersymmetry at the surface. Through this mapping, we are able to calculate the surface multifractal exponents, $\Delta_s^2$ and $\Delta_s^3$, of the wavefunctions. In addition, we also extract the surface scaling exponents of thermodynamic and transport quantities.
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Outline

• L-D transitions and Disordered Electronic Systems.

• Surface Criticality and Multifractality.

• Analyzing The Spin Quantum Hall Transition using SUSY.

• Calculation of Surface Exponents.

• Summary.
Localization-Delocalization (LD) Transitions

Metal ↔ Insulator - induced by disorder.
No obvious broken symmetry.
No formal order parameter.

Examples:
Anderson Transition
Integer Quantum Hall Effect.
Integer Quantum Hall Effect

Electrons in 2D – strong $\perp \vec{B}$ field.

Plateau Transition Theory?

i. Pruisken’s Non-linear Sigma Model

ii. Chalker-Coddington Network Model

iii. Antiferromagnetic Superspin Chain (N. Read)

(Paalonen, Tsui, Gossard 1982)
## Disordered Electronic Systems

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<th>Symmetry Class</th>
<th>Time Reversal</th>
<th>Spin Rotation</th>
<th>Additional Symmetry</th>
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</table>

(Altland and Zimbrauer 1997)

IQHE  
Anderson Localization  
Spin Quantum Hall  
(Senthil et al 1998)
Multifractality

\[ \left\langle |\psi(r)|^{2q} \right\rangle \sim L^{-\tau_q - d} \]

\[ \tau_q = d(q - 1) + \Delta_q \]

Metal: \( \Delta_q = 0 \)

Insulator: \( \tau_q + d = 0 \)

Multifractal Wavefunction Intensity

from Huckestein
Surface Criticality

- Surface breaks translation symmetry.
- When correlation length diverges, large scale geometry of the system is important.
- Even mean field theory gives different exponents in the presence of a boundary.

Eg. Bulk: \( G(r_1, r_2) \sim r_1^{d+2} r_2^{-d} \)

Surface: \( G(r_1, r_2) \sim r_1^{-d} r_2^{d+2} \)
SQH Network Model

Class C Network models:

Hamiltonian SU(2) invariant.

$U^\dagger \sigma_y U = \sigma_y \rightarrow \text{Sp}(2n)$ scattering matrices. Our case $n = 2$.

Polar Decomposition:

$$S = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} \sqrt{1-T^2} & T \\ -T & \sqrt{1-T^2} \end{pmatrix} \begin{pmatrix} U_3 & 0 \\ 0 & U_4 \end{pmatrix}$$

(Kagalovsky 1999)
Double links ($\sigma = \uparrow, \downarrow$) connecting nodes of two kinds, A and B.

Nodes diagonal in spin indices. All A (B) nodes have same scattering matrices.

Disorder only on links $\rightarrow$ random SU(2).

$$t_{A\sigma}^2 + t_{B\sigma}^2 = 1 \rightarrow \text{isotropic network.}$$

$$t_{A\sigma} = t_{B\sigma} : \text{critical point.}$$

$$t_{\uparrow} = t_{\downarrow} : \text{SQH}$$
Second Quantized Description

Each uplink (downlink):
2 sets of bosons \( b_\sigma b_\sigma^\dagger \) \( \widetilde{b}_\sigma \widetilde{b}_\sigma^\dagger \)
2 sets of fermions \( f_\sigma f_\sigma^\dagger \) \( \widetilde{f}_\sigma \widetilde{f}_\sigma^\dagger \)
(\( \sigma = \uparrow \) and \( \downarrow \))

Key Point:
No separate operators for advanced Green’s functions
\( G_A = - (\varepsilon R \varepsilon)^T \)

(Gruzberg, Ludwig & Read 1999)
\[ B = \frac{1}{2}(b_\uparrow \uparrow b_\uparrow + b_\downarrow \downarrow b_\downarrow + 1), \quad Q_3 = \frac{1}{2}(f_\uparrow \uparrow f_\uparrow + f_\downarrow \downarrow f_\downarrow - 1) \]

\[ Q_+ = f_\uparrow \uparrow f_\downarrow \downarrow, \quad Q_- = f_\downarrow \downarrow f_\uparrow \uparrow, \]

\[ V_+ = \frac{1}{\sqrt{2}}(b_\uparrow \uparrow f_\downarrow \downarrow - b_\downarrow \downarrow f_\uparrow \uparrow), \quad W_- = (V_+)\dagger, \]

\[ V_- = -\frac{1}{\sqrt{2}}(b_\uparrow \uparrow f_\uparrow \uparrow + b_\downarrow \downarrow f_\downarrow \downarrow), \quad W_+ = -(V_-)\dagger. \]
Supersymmetry

\[(J_{2i-1} + \bar{J}_{2i})V_{2i-1,2i} = V_{2i-1,2i}(J_{2i-1} + \bar{J}_{2i})\]
\[(\bar{J}_{2i} + J_{2i+1})V_{2i,2i+1} = V_{2i,2i+1}(\bar{J}_{2i} + J_{2i+1})\]

SURFACE:

Reflecting boundary conditions: still full supersymmetry
Disorder Averaging

SU(2) averaging projects infinite dimensional Fock space of bosons and fermions onto the fundamental (dual) 3 dimensional representation of sl (2|1) on the uplinks (downlinks).

Percolation Mapping

\[ \pi\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \pi\left(-\frac{1}{2}, \frac{1}{2}\right) = \pi(0,0) + \pi(0,1) \]

\[ [V_{12}] = t_A^2 P_s + (1 - t_A^2)I \otimes I \]

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\[ [V_{23}] = t_B^2 P_s + (1 - t_B^2)I \otimes I \]

\[ V_{12} = t_A^2 + (1 - t_A^2) \]

\[ V_{23} = t_B^2 + (1 - t_B^2) \]
Equivalent Mapping:

- Q=1 Potts Model.

Two relevant scaling exponents:

- $x_{1b} = \frac{1}{4}$
- $x_{1s} = \frac{1}{3}$
Physical Quantities

Green’s Function:

\[ G_{\alpha \beta}(r_1, r_2) = S \text{Tr} \ b_\alpha(r_1) b_\beta^\dagger(r_2) U \]

Strategy:

Express physical quantities in terms of Green’s functions, then write these as disorder averaged \text{sl}(2\|1)\ generators.

Evaluate in the fundamental representation to find equivalent percolation probabilities.
Thermodynamic Exponents

Density of States:

\[ \langle \rho(r_1, \epsilon) \rangle = \frac{1}{4\pi} \langle \text{Tr} G_R(r_1, r_1, z) - \text{Tr} G_A(r_1, r_1, z) \rangle \]

\[ = \frac{1}{2\pi} \langle \text{Tr} G_R(r_1, r_1, z) - 1 \rangle \]

\[ = \frac{1}{2\pi} \langle 2B(r_1) \rangle \]

\[ = \frac{1}{2\pi} \left[ 1 - \sum_N P(r_1; N) \cos 2N\epsilon \right] \]

Bulk:

\[ \rho(\epsilon) \propto \epsilon^{x^b_1/(2-x^b_1)} = \epsilon^{1/7} \]

Surface:

\[ \rho(r_1, \epsilon) \propto \epsilon^{x^s_1/(2-x^b_1)} = \epsilon^{4/21} \]
Transport Exponents

Diffusion Propagator:
\[ \langle \Pi(r_1, r_2) \rangle = \langle \text{Tr} \left[ G_R(r_1, r_2) G_A(r_2, r_1) \right] \rangle = -2 \langle V_-(r_1) W_+(r_2) \rangle \]

Point-Contact Conductance:
\[ \langle g_{\text{point}}(r_1, r_2) \rangle = \langle f_{\uparrow}^\dagger(r_1) f_{\downarrow}^\dagger(r_1) f_{\downarrow}(r_2) f_{\downarrow}(r_2) \rangle = \langle Q_+(r_1) Q_-(r_2) \rangle \]

\( r_1 \) and \( r_2 \) in bulk:
\[ \sim r^{-2x_{1b}} \sim r^{-1/2} \]

\( r_1 \) and \( r_2 \) on surface:
\[ \sim r^{-2x_{1s}} \sim r^{-2/3} \]

\( r_1 \) in bulk and \( r_2 \) on surface:
\[ \sim r^{-x_{1b} - x_{1s}} \sim r^{-7/12} \]
Multifractal Exponents

\[ \langle |\psi(r)|^{2q} \rangle \sim L^{-\tau_q - d} \]

\[ \tau_q = d(q - 1) + \Delta_q \]

Away from criticality:

\[ L^{2q} \langle |\psi(r_1)|^2 |\psi(r_2)|^2 \cdots |\psi(r_q)|^2 \rangle \sim \left( \frac{r}{\xi_\epsilon} \right)^{\Delta_q} \quad r \leq \xi_\epsilon \]
(2\pi)^2 \tilde{D}^2 (r_1, r_2)

= \left\langle \sum_{ij\alpha\beta} |\psi_{i\alpha}(r_1)|^2 |\psi_{j\beta}(r_2)|^2 \delta(\epsilon_1 - \epsilon_i)\delta(\epsilon_2 - \epsilon_j) \right\rangle

= 4 \sum_N \left[ 1 - z^{2N} \right] P(r_1, r_2, ; N)

+ 4 \sum_{N,N'} \left[ 1 - z^{2N} \right] \left[ 1 - z^{2N'} \right] \left[ P_-(r_1, r_2; N, N') \right]

(Mirlin, Evers & Mildenberger 2003)

$\Delta_2$ calculation: $z = e^{-\gamma} \rightarrow 1$: exact cancellation.

$P(r_1, r_2; N) \sim N^{-25/21}r^{-1/3}$, \quad $r \equiv |r_1 - r_2| \lesssim N^{4/7}$. 
Multifractal Exponents

$\Delta_2 = -1/4$  $\Delta_2 = -1/4$  $\Delta_2 = -1/3$

$\Delta_3$ calculation: surprising!

$\Delta_3 = -3/4$  $\Delta_3 = -5/6$  $\Delta_3 = -11/12$  $\Delta_3 = -1$
Network Simulation

$\Delta_2$ – 0.5 % deviation

$\Delta_3$ - 6 % deviation

(Evers, Mildenberger, Mirlin)
Summary

• New paradigm – Criticality and Multifractality at the surface in LD transitions.
• Illustration in the spin quantum Hall case.
Current Work

- Surface behavior in other LD transitions, esp. Integer Quantum Hall.
- Understanding surface multifractality.
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